CS, Econ, Math and Everything in Between

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Chapter-1: Computer Science

Go and explain your daughter that you are proving LOWER BOUNDs on the speed at which a computer can solve a given problem. Pull your hair crazily when she asks why you would want to do such a thing: don't you want the computer to be as fast as possible?

The Pen-Testing Problem [Qiao and Valiant 2023]

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The Pen-Testing Problem [Qiao and Valiant 2023]

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- Ink level v_i of pen i comes from distribution D_i
- Want to pick a pen with **largest residual ink level**

- Write pen i for time t_i
- If $v_i > t_i$, we know pen i had more than t_i units of ink at the start
- If $v_i \leq t_i$, we know pen i has run out of ink

- Ink is irrevocably used up while testing

Applications- Light Bulbs

- Light bulbs either good or faulty
- Faulty bulbs tend to go off within a day or two
- Good bulbs last longer than a month
- Choose best 100 out of 300 bulbs?

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- Leave bulbs on for two days
  - Faulty bulbs fuse by then
  - Good bulbs lose two days of their lifetime

Example

- 2 pens
  - 30 minutes ink wp ½
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- Average ink
  - $\frac{1}{2} \times \frac{1}{2} \times 0 + (1 - \frac{1}{2} \times \frac{1}{2}) \times 29 = 21.75$

Fancier Feasibility Constraints

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  - Pens have sizes and ink levels
  - Can pick any set of pens as long as they fit your pouch

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- Generally:
  - A feasibility set F of subsets of pens
  - Can choose any set of pens from F
  - Eg: {{1, 2, 4}, {1, 3, 4}, {2, 3, 4}}

Application- Team Selection

- Select a cricket team
  - 6 batters/4 bowlers/1 all-rounder
  - 3 pacers/2 spinners
  - etc

- Select team by judging performance in domestic games and net sessions
  - Too many sessions lead to burnt-out players and bad performance for the team

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Back to Example

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- Select 1 pen

- We got: 21.75
- God:
  - Wp ½ x ½: both pens have 1 minute ink
  - Wp (1 - ½ x ½): at least 1 pen has 30 minutes ink
  - Average: ½ x ½ x 1 + (1 - ½ x ½) x 30 = 22.75

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- We are 21.75/22.75 ≅ 0.956 fraction of god

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- Want to say Us/God ≥ ? for any distribution $D_1 x D_2 x \dots x D_n$
- Residual ink/Original ink

Example-2: The Exponential Distribution

- 2 pens
  - Ink levels from the exponential distribution
- Select 1 pen

- Exponential distribution:
  - $Pr(Q > t) = e^{-\lambda t}$

Exponential Distribution at λ = 1

- $\Pr(Q > t) = e^{-\lambda t}$

- Quantity always greater than zero
- Approaches zero probability as quantity goes to infinity

Memorylessness of the Exponential Distribution

- $Pr(Q > T+t \mid Q > T)$- what is the probability of at least T+t units of ink, if promised there is at least T units of ink?

- Why care about such a question?
  - Exactly captures burnt-ink
  - Is there t more units of ink after using up T units of ink

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- $\Pr(Q > T+t \mid Q > T)$- what is the probability of at least T+t units of ink, if promised there is at least T units of ink?

- $\Pr(Q > T+t) = e^{-\lambda(T+t)}$
- Normalize, so that $\Pr(Q > T) = 1$

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$$= e^{-\lambda(T+t)} / e^{-\lambda T} = e^{-\lambda t}$$

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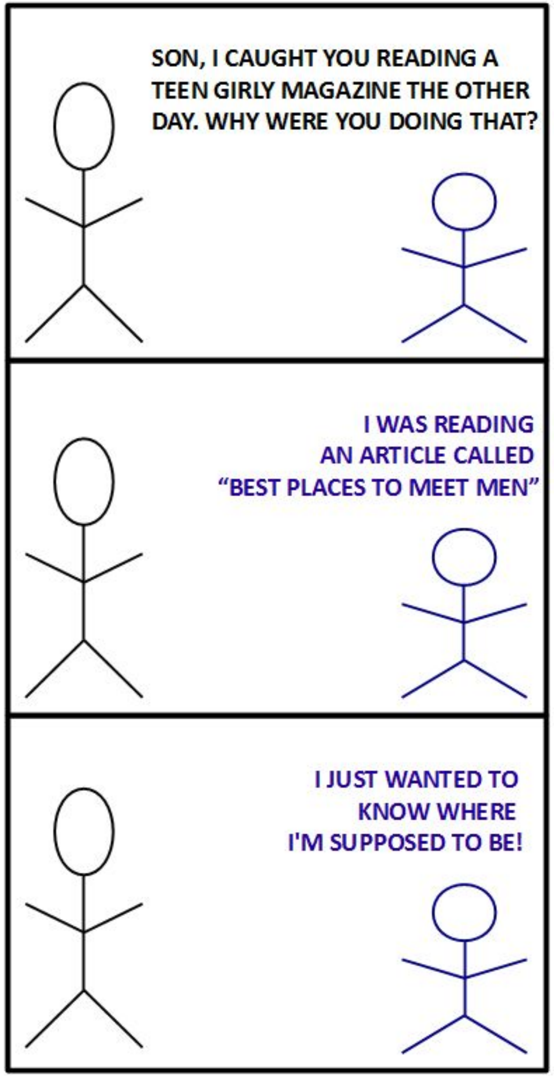
- $Pr(Q > T+t \mid Q > T)$- what is the probability of at least T+t units of ink, if promised there is at least T units of ink?

- $Pr(Q > T+t \mid Q > T) = e^{-\lambda t}$
  - Equals $Pr(Q > t)$

- Even if ink has not dried up after T units of time, we learn nothing about the distribution of remaining ink!
  - Testing gives very little information

Chapter-2: Economics



Surplus Maximization

- Selling a litre of petrol
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- Can allocate anyone at most half a litre of petrol
  - Nobody has a one litre bottle

Surplus Maximization

- Surplus = Total utility for selling the litre of petrol

- Eg: Sell all three of us ⅓ litres of petrol and charge each of us Rs 110 per litre.
    - My utility: ⅓ x (100-110) = -3.333
    - Driver's utility: ⅓ x (120-110) = 3.333
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- Surplus = ⅓ x 100 + ⅓ x 120 + ⅓ x 135

Surplus Maximization

- My value: Rs 100, Driver's value: Rs 120, Ambani's value: Rs 135
- Can't sell more than half a litre of petrol to someone

- How to optimize surplus? Interesting for a Government, for eg.
  - Give half a litre each to Driver and Ambani
  - Charge everyone nothing

- More generally, optimize total value and charge nothing

Challenge

- I am angry at being left out
- I am also sneaky- when auctioneer asks me my value:
  - Answer 100 and be left out
  - Answer 200 and get half a litre petrol

- My happiness
  - $\frac{1}{2} \times (200 - 0) = 100$

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- Need to optimize surplus even when bidders are strategic

Ascending Price Auction (IPL Auction)

- Two identical items
    - Initially price = 0, everybody has hands up
    - Price keeps increasing bit by bit
    - Bidders drop down hands when price crosses their value
    - Stop increasing price until there are only two hands up

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  - Price keeps increasing bit by bit
  - Bidders drop down hands when price crosses their value
  - Stop increasing price until there are only two hands up

- Give item to the two people with hands up
- Charge the last price from the two people

What Should Bidders Do?

- Stop before price reaches their value?
  - No: still profitable if the auction ends

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- Keep going after price crosses their value
  - No: will pay more than value if they end up winning

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- Keep going after price crosses their value
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- Optimal strategy to keep hands up until your price reaches your value and drop hands after

Funky Ascending “Hands Up” Auction- Deferred Acceptance Auctions [Milgrom and Segal 2014]

- Need not increase price uniformly for all bidders

- Odd numbered rounds- increase price for odd numbered agents by ϵ
- Even numbered rounds- increase price for even numbered bidders by 2ϵ

- What property does such an auction satisfy?
  - Who knows? But, why not?

Consumer Surplus

- Part of surplus that comes from consumers

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- My value: Rs 100, Driver's value: Rs 120, Ambani's value: Rs 135
- Charge Driver and Ambani Rs 110 per litre and allocate both half a litre
  - My utility: 0
  - Driver's utility: $\frac{1}{2} \times (120-110) = 5$
  - Ambani's utility: $\frac{1}{2} \times (135-110) = 12.5$

- Consumer surplus = 5 + 12.5 = 17.5

Why is Consumer Surplus Interesting?

- What if the payment cannot be transferred to the auctioneer?
  - What is bidders make payment in time? Wait in line for an hour to get access
  - Time cannot be transferred to the auctioneer

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  - Charge packets while routing them on the internet by slowing down delivery [Hartline and Roughgarden 2008]

Consumer Surplus and Pen Testing [AG and Hartline 2023]

Auction Quantity	Analogous Pen Testing Quantity
Surplus	Original ink quantity
Payment	Ink Burnt
Consumer Surplus = Surplus - Revenue	Residual Ink = Original ink quantity - ink burnt

- Ascending “hands up” prices similar to burning ink drop by drop

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- Pen testing same as designing consumer surplus optimizing “hands up” auction!

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- Pen testing same as designing consumer surplus optimizing “hands up” auction!
- Residual ink/original ink = optimal consumer surplus/optimal surplus

Chapter-3: Mathematics

A mathematician is asked by a friend who is a devout Christian: "Do you believe in one God?" He answers: "Yes - up to isomorphism."

What do we want to Solve?

$\max_{(x, p)} \min_{\text{all distributions } D}$ avg. consumer surplus/ optimal avg. surplus

such that

- Truthful bidding is the optimal strategy for all bidders i

Truthful Bidding

- $x_i(.)$ be the average allocation to bidder i, averaged over the values of other bidders
- $p_i(.)$ be the average payment of bidder i, averaged over the values of the other bidders

- We want user with value v to bid v and not b.

$$v.x_i(v) - p_i(v) \geq v.x_i(b) - p_i(b)$$

What do we want to Solve?

$\max_{(x,\, p)} \min_{\text{all distributions } D}$ avg. consumer surplus/ optimal avg. surplus

such that

- $v.x_i(v) - p_i(v) \geq v.x_i(b) - p_i(b)$
  - for all v, b, i

Avg. Surplus

- Let y be the surplus optimal mechanism
- Contribution of bidder i to the surplus when value is v
  - $v.y_i(v)$
  - Avg. surplus from i = $\sum_v \Pr(v_i = v) \times v.y_i(v)$

- Avg. surplus = $\sum_i \sum_v \Pr(v_i = v) \times v.y_i(v)$

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avg. consumer surplus/ $\sum_i \sum_v Pr(v_i = v) \times v.y_i(v)$

such that

- $v.x_i(v) - p_i(v) \geq v.x_i(b) - p_i(b)$
  - for all v, b, i
- y is the surplus optimal mechanism

Avg. Consumer Surplus

- Consumer surplus = surplus - payment
- Avg. Consumer surplus = avg. surplus - avg. payment

Avg. consumer surplus

$$= \sum_i (\sum_v Pr(v_i = v) \times v.x_i(v) - \sum_v Pr(v_i = v) \times p_i(v))$$

$$= \sum_i \{\sum_v Pr(v_i = v) \times [v.x_i(v) - p_i(v)]\}$$

What do we want to Solve?

$\max_{(x,\, p)} \min_{\text{all distributions } D}$

$$\sum_i \{\sum_v \Pr(v_i = v) \times [v.x_i(v) - p_i(v)]\} \,/\, \sum_i \sum_v \Pr(v_i = v) \times v.y_i(v)$$

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For all Distributions D

- $\sum_v \Pr(v_i = v) = 1$
- $\Pr(v_i = v) \geq 0$

What do we want to Solve?

$\max_{(x,\, p)} \min_{i,\, Pr(vi = \,)}$

$$\sum_i \{\sum_v Pr(v_i = v) \times [v.x_i(v) - p_i(v)]\} / \sum_i \sum_v Pr(v_i = v) \times v.y_i(v)$$

such that

- $v.x_i(v) - p_i(v) \geq v.x_i(b) - p_i(b)$
  - for all v, b, i
- y is the surplus optimal mechanism
- $\sum_v Pr(v_i = v) = 1$, $Pr(v_i = v) \geq 0$

y is the surplus optimal mechanism

- Suppose the optimal average surplus is 5
  - What does that mean?

- $5 \geq \sum_i \sum_v Pr(v_i = v) \times v.y_i(v)$
  - for all allocation rules y

- Clearly
  - $6 \geq \sum_i \sum_v Pr(v_i = v) \times v.y_i(v)$
  - $7 \geq \sum_i \sum_v Pr(v_i = v) \times v.y_i(v)$ and so on

y is the surplus optimal mechanism

- Suppose the optimal average surplus is 5
  - What does that mean?

- $5 \geq \sum_i \sum_v \Pr(v_i = v) \times v.y_i(v)$
  - for all allocation rules y

- 5 is the smallest real number t such that

$$t \geq \sum_i \sum_v \Pr(v_i = v) \times v.y_i(v) \text{ for all allocation rules } y$$

y is the surplus optimal mechanism

- Suppose the optimal average surplus is t
    - What does that mean?

- min t such that
    - $t \geq \sum_i \sum_v Pr(v_i = v) \times v.y_i(v)$ for all allocation rules y

What do we want to Solve?

$\max_{(x,\, p)} \min_{i,\, Pr(vi\, =\,),\, t}$

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such that

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- $\sum_v Pr(v_i = v) = 1,\ Pr(v_i = v) \geq 0$

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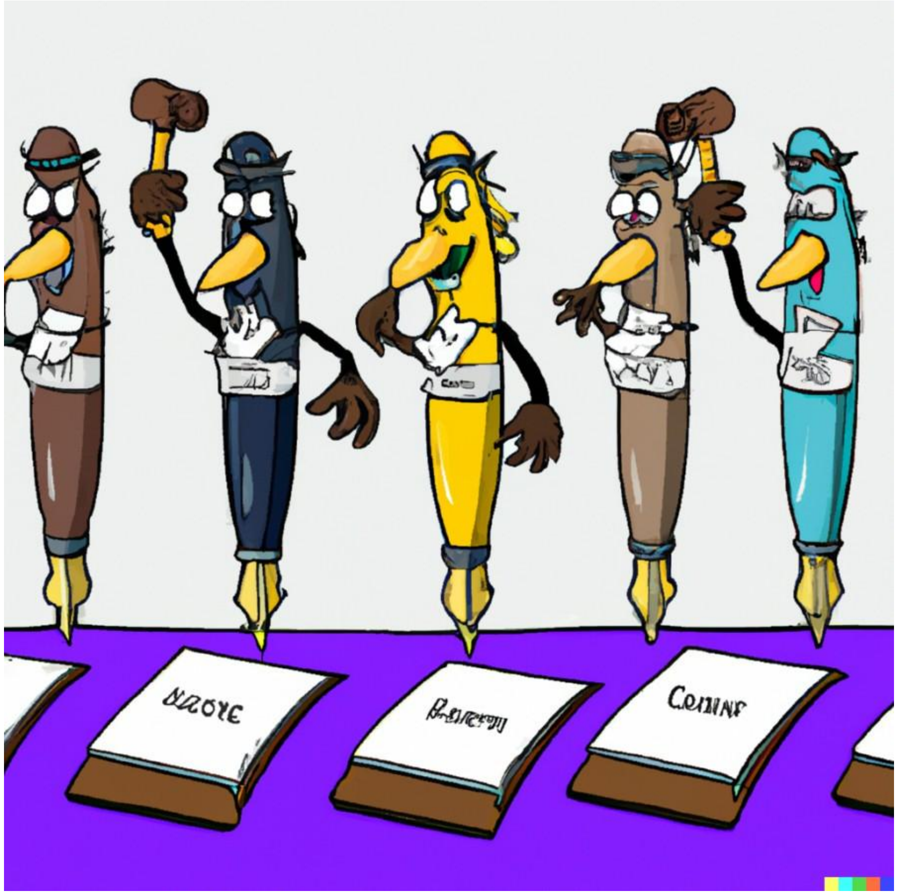
- $\sum_v Pr(v_i = v) = 1$, $Pr(v_i = v) \geq 0$

An optimization problem with a bunch of functions as constraints- ready for calculus on steroids!

Turns out [AG and Hartline 2023]

- Residual ink/Original ink:
  - c log n for some constant c for knapsack, k out of n pens
  - $\log^2 n$ for more general (downward closed) constraints

Takeaway



Thank you!

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